ELIXIIT ACADEMY

— The Toppers' Point

IIT- JEE MONTHLY MOCK TEST→ II (Solutions)

Solutions to Paper→1: OBJECTIVE & NUMERICAL TYPE QUESTIONS

1. We have $\lim_{x\to 2}\frac{f(x)g(x)}{f'(x)g'(x)} = 1$

$\Rightarrow\ \lim_{x\to 2}\frac{f'(x)g(x)+f(x)g'(x)}{f''(x)g'(x)+f'(x)g''(x)} = 1$ (Using L Hospital's Rule)

$\Rightarrow \frac{f'(2)g(2)+f(2)g'(2)}{f''(2)g'(2)+f'(2)g''(2)} = 1$ [Since f', f'', g', g'' are continuous]

$\Rightarrow \frac{f(2)\,g'(2)}{f''(2)g'(2)} = 1$ [Since, $f'(2) = g(2) = 0$]

$\Rightarrow f''(2) = f(2)$

$\Rightarrow f''(2) > 0$ [Since $f: R \to (0, \infty)$ $\therefore f(2) > 0$]

Thus, we have $f'(2) = 0$ and $f''(2) > 0$. So, f has a local minimum at $x = 2$.

Again, $f''(2) = f(2)$

$\Rightarrow f(2) - f''(2) = 0$

$\Rightarrow f(x) - f''(x) = 0\ at\ x = 2$

$\Rightarrow f(x) - f''(x) = 0\ for\ at\ least\ one\ x \in R.$

Option (b) and (d) are correct.

Q2. $x^{2x} - 2x^x \cot y - 1 = 0$ …….. (1)

When $x = 1$, we get

$11 - 2\cot y - 1 = 0 \Rightarrow \cot y = 0 \Rightarrow y = \frac{\pi}{2}$

Differentiating (1) w.r.t x, we get

$$2x^{2x}(1 + \log x) - 2x^x(1 + \log x)\cot y + 2x^x cosec^2 y \frac{dy}{dx} = 0$$

Putting $x = 1, y = \frac{\pi}{2}$, we get

$$2 - 2 \times 0 + 2\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -1$$

Q3. Consider the function $\phi(x)$ given by

$$\phi(x) = \int_0^x (1 + \cos^8 t)(at^2 + bt + c)\, dt$$

$$\Rightarrow \phi'(x) = (1 + \cos^8 x)(ax^2 + bx + c) \qquad \text{………… (1)}$$

We observe that

$$\phi(0) = 0$$

$$\phi(1) = \int_0^1 (1 + \cos^8 t)(at^2 + bt + c)\, dt = 0 \quad (Given)$$

$$and\ \phi(2) = \int_0^2 (1 + \cos^8 t)(at^2 + bt + c)\, dt = 0 \quad (Given)$$

Therefore, 0, 1 and 2 are the roots of $\phi(x)$.

By Rolle's Theorem $\phi'(x) = 0$ will have at least one real root between 0 and 1 and at least one real root between 1 and 2.

Q4. Let $P(4t^2+3, 8t^3-1)$ be a point on the given curve.

Now, $x = 4t^2+3, y = 8t^3-1$

$\Rightarrow \frac{dx}{dt} = 8t \;\; and \;\; \frac{dy}{dt} = 24t^2$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{24t^2}{8t} = 3t$

The tangent to the curve at P is

$$y-(8t^3-1) = 3t(x-4t^2-3)$$

This will pass through $Q(4t_1^2+3, 8t_1^3-1)$, if

$(8t_1^3-1)-(8t^3-1) = 3t(4t_1^2+3-4t^2-3)$

$\Rightarrow 8(t_1^3-t^3) = 12t(t_1-t^2)$

$\Rightarrow 2(t_1^2+tt_1+t^2) = 3t_0(t_1+t)$

$\Rightarrow 2t_1^2 - t_1t - 2t^2 = 0$

$\Rightarrow (t_1-t)(2t_1+t) = 0$

$\Rightarrow t_1 = -\frac{t}{2}$ $\qquad\qquad$ $[Since,\ t_1 \neq t]$

Option (a) is correct.

Q5. We have,

$$f'(1) = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}, f'(2) = \tan\frac{\pi}{3} = \sqrt{3}\ and\ f'(3) = \tan\frac{\pi}{4} = 1.$$

$$\therefore \int_2^3 f'(x)f''(x)dx + \int_1^3 f''(x)\,dx$$

$$= \int_2^3 f'(x)\, d\big(f'(x)\big) + \int_1^3 d\big(f'(x)\big)$$

$$= \left[\frac{\{f(x)\}^2}{2}\right]_{x=1}^{x=3} + [f'(x)]_{x=1}^{x=3}$$

$$= \frac{1}{2}[\{f'(3)\}^2 - \{f'(2)\}^2] + [f'(3) - f'(1)]$$

$$= \frac{1}{2}(1-3) + \left(1 - \frac{1}{\sqrt{3}}\right) = -\frac{1}{\sqrt{3}}$$

Q6. Length of the arc $= a\theta$

$\implies 2\pi r = a\theta$ (where r is the radius of the base of the funnel)

$$\implies r = \frac{a\theta}{2\pi}$$

$$\implies h = height\ of\ the\ funnel = \sqrt{a^2 - r^2} = a\sqrt{\left\{1 - \left(\frac{\theta}{2\pi}\right)^2\right\}}$$

$$\text{Volume (V)} = \frac{1}{3}\pi\left(\frac{a\theta}{2\pi}\right)^2 a\sqrt{1 - \left(\frac{\theta}{2\pi}\right)^2}$$

Evaluate $\frac{dV}{d\theta}$ and equate that to zero to find the value of θ. After equating to zero your equation will be,

$$\implies 2 - 2\left(\frac{\theta}{2\pi}\right)^2 - \left(\frac{\theta}{2\pi}\right)^2 = 0$$

$$\implies \frac{\theta}{2\pi} = \sqrt{\frac{2}{3}}$$

$$\implies \theta = 2\pi\sqrt{\frac{2}{3}}$$

Option (b) is correct.

Q7. Here $\frac{dy}{dx} = \frac{2(1-x^2)}{(1+x^2)^2}$

Again $\frac{d^2y}{dx^2} = \frac{4x(x^2-3)}{(1+x^2)^2}$

Now $\frac{d^2y}{dx^2} = 0$ gives, three solutions, $x = 0,\ x = \pm\sqrt{3}$

$\frac{dy}{dx} = 0$, gives, $x = \pm 1$ at which $\frac{d^2y}{dx^2} < 0$ for $x = 1$ and > 0 for $x = -1$ i.e. maximum and minimum points.

So, option (a) is correct.

Q8. Now, $x^2 - 2x - 3 > 0$

$\Rightarrow (x-1)^2 > 4$

$\Rightarrow x - 1 > 2\ \ or\ x - 1 < -2$

$\Rightarrow x > 3\ \ or\ \ x < -1$

$$f(x) = \frac{\log(x^2 - 2x - 3)}{\log\left(\frac{1}{2}\right)} = -\frac{\log(x^2 - 2x - 3)}{\log 2}$$

$$f'(x) = -\frac{1}{\log 2}\left(\frac{2x - 2}{x^2 - 2x - 3}\right)$$

$$f'(x) = -\left(\frac{1}{\log 2}\right)\frac{2(x-1)}{\{(x+1)(x-3)\}}$$

To be monotonically increasing $f'(x)$ must be > 0

Let us take $x = 4$, we $f'(4) < 0$

$\Rightarrow$ Options (c) and (d) cannot be true.

Now, let us put $x = 0$, we get, $f'(0) < 0$

$\Longrightarrow$ Option (a) cannot be true.

Q9. $\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{a_0+a_1|x|+a_2|x|^2+a_3|x|^3-a_0}{x}$

$= \lim_{x\to 0^-} \left(\frac{-a_1x+a_2x^2-a_3x^3}{x}\right) = \lim_{x\to 0^-}(-a_1 + a_2x - a_3x^2) = -a_1$

$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{a_0+a_1|x|+a_2|x|^2+a_3|x|^3}{x}$

$= \lim_{x\to 0^+} \left(\frac{a_1x+a_2x^2+a_3x^3}{x}\right) = \lim_{x\to 0^+}(a_1 + a_2x + a_3x^2) = a_1$

Now, if $f(x)$ is differentiable at $x = 0$ then $-a_1 = a_1$ $i.e., a_1 = 0$

Option (c) is correct.

Q10. If f(x) is continuous at $x = 1$, then

$\lim_{x\to 1} f(x) = f(1)$

$\Longrightarrow \lim_{x\to 1} f(x) = k$

$\Longrightarrow \lim_{x\to 1} \frac{x^{2^{32}}-2^{32}x+4^{16}-1}{(x-1)^2} = k$

$\Longrightarrow \lim_{x\to 1} \frac{x^n-nx+n-1}{(x-1)^2} = k\ , where\ n = 2^{32}$

$\Longrightarrow \lim_{x\to 1} \frac{(x^n-1)-n(x-1)}{(x-1)^2} = k$

$\Longrightarrow \lim_{x\to 1} \frac{\frac{x^n-1}{x-1}-n}{x-1} = k$

$\Longrightarrow \lim_{x\to 1} \frac{(x^{n-1}+x^{n-2}+\cdots+x+1)-n}{x-1} = k$

$\Longrightarrow \lim_{x\to 1} \frac{(x^{n-1}-1)+(x^{n-2}-1)+\cdots+(x-1)}{x-1} = k$

$\Longrightarrow \lim_{x\to 1}\left(\frac{x^{n-1}-1}{x-1}+\frac{x^{n-2}-1}{x-1}+\cdots+\frac{x^2-1}{x-1}+\frac{x-1}{x-1}\right)=k$

$\Longrightarrow (n-1)+(n-2)+\cdots+2+1=k$

$\Longrightarrow k=\frac{n(n-1)}{2}=\frac{2^{32}(2^{32}-1)}{2}=2^{63}-2^{31}$

NUMERICAL ANSWER TYPE (NAT) QUESTIONS:

Q11. Let $P=(2\sqrt{2}\cos\theta\,,3\sqrt{2}\sin\theta)$

Now, $\frac{x^2}{8}+\frac{y^2}{18}=1$

$\Longrightarrow \frac{x}{4}+\left(\frac{y}{9}\right)\left(\frac{dy}{dx}\right)=0$

$\Longrightarrow \frac{dy}{dx}=-\frac{9x}{4y}$

$\Longrightarrow \left(\frac{dy}{dx}\right)_P=-\frac{9\times 2\sqrt{2}\cos\theta}{4\times 3\sqrt{2}\sin\theta}=-\frac{3\cot\theta}{2}$

Equation of AB is, $y-3\sqrt{2}\sin\theta=\frac{-3\cot\theta}{2}(x-2\sqrt{2}\cos\theta)$

Putting y = 0 we get, $x=\frac{2\sqrt{2}\cos^2\theta+2\sqrt{2}\sin^2\theta}{\cos\theta}=\frac{2\sqrt{2}}{\cos\theta}$

Putting $x=0$ we get, $y=\frac{3\sqrt{2}}{\sin\theta}$

Therefore, $A=\left(\frac{2\sqrt{2}}{\cos\theta},0\right)$ and $B=\left(0,\frac{3\sqrt{2}}{\sin\theta}\right)$

Therefore, area of triangle OAB=S=$\frac{1}{2}\times\frac{2\sqrt{2}}{\cos\theta}\times\frac{3\sqrt{2}}{\sin\theta}=\frac{12}{\sin 2\theta}$

$\frac{dS}{d\theta}=\left(-\frac{12}{\sin^2 2\theta}\right)(2\cos 2\theta)=0\Longrightarrow\theta=\frac{\pi}{4}$

Area $=\frac{12}{\sin\frac{\pi}{2}}=12.$

Q12. Using L Hospital's Rule, we have

$$\lim_{h\to 0}\frac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)}=\lim_{h\to 0}\frac{f'(2h+2+h^2)\times(2+2h)-0}{f'(h-h^2+1)(1-2h)-0}=\frac{f'(2)\times 2}{f'(1)}=\frac{12}{4}=3$$

Q13. The vertex of the parabola $y = ax^2 + bx + c$ is at $\left(-\frac{b}{2a}, -\frac{b^2-4ac}{4a}\right)$

$\therefore -\frac{b}{2a} = 4 \;\; and \;\; -b^2 - \frac{4ac}{4a} = \; 2$

$\Longrightarrow b = -8a \quad and \quad c - \frac{b^2}{4a} = 2$

$\Longrightarrow b = -8a \quad and \quad c = 16a + 2$

Now,

$f(a) = abc = -8a^2(16a + 2) = -16(8a^3 + a^2)$

$\frac{df(a)}{da} = -16(24a^2 + 2a)$

Clearly,

$\frac{df(a)}{dz} < 0$ for all $a \in [1, 3]$

$\Longrightarrow$ f(a) is decreasing on [1, 3]

$\therefore$ Minimum Value of f(a) = f(3)=$-16(8 \times 3^3 + 3^2) = -3600$

Maximum Value of f(a)=f(1)=$-16(8 + 1) = -144$

Hence, the required difference = $-144 + 3600 = 3456$

Q14. Let $\alpha_r = \frac{x}{2^{r+1}}, r = 0,1,2, \dots, n$. Then,

$$\frac{\tan\frac{x}{2^{r+1}}+\tan^3\frac{x}{2^{r+1}}}{1-\tan^2\frac{x}{2^{r+1}}}$$

$= \frac{\tan\alpha+\tan^3\alpha_r}{1-\tan^2\alpha_r} = \tan\alpha_r\left(\frac{1+\tan^2\alpha_r}{1-\tan^2\alpha_r}\right) = \frac{\tan\alpha_r}{\cos 2\alpha_r} = \frac{\sin\alpha_r}{\cos\alpha_r cos2\alpha_r} = \frac{\sin(2\alpha_r-\alpha_r)}{\cos\alpha_r\cos 2\alpha_r}$

$= \tan 2\alpha_r - \tan\alpha_r$

$$\therefore f(x) = \lim_{n\to\infty}\sum_{r=0}^{n}(\tan 2\alpha_r - \tan\alpha_r)$$

$$\Rightarrow f(x) = \lim_{n\to\infty}\sum_{r=0}^{n}\left(\tan\frac{x}{2^r} - \tan\frac{x}{2^{r+1}}\right)$$

$$\Rightarrow f(x) = \lim_{n\to\infty}\left(\tan x - \tan\frac{x}{2^{n+1}}\right) = tanx$$

Hence, $\lim_{x\to 0}\frac{f(x)}{x} = \lim_{x\to 0}\frac{\tan x}{x} = 1$

Q15. It is given that the polynomial f(x) satisfies the relation

$$\left(f(\alpha)\right)^2 + \left(f'(\alpha)\right)^2 = 0$$

$$\therefore f(\alpha) = 0 = f'(\alpha)$$

$\Rightarrow x = \alpha$ is the root of $f(x)\ and\ f'(x)$

$\Rightarrow (x-\alpha)^2$ is a factor of f(x)

Let $f(x) = (x-\alpha)^2\phi(x)$. Then,

$f'(x) = 2(x-\alpha)\,\phi(x) + (x-\alpha)^2\,\phi'(x)$

$$\therefore \frac{f(x)}{f'(x)} = \frac{(x-\alpha)\,\phi(x)}{2\phi(x) + (x-\alpha)\,\phi'(x)}$$

Now,

$\lim_{x\to\alpha}\frac{f(x)}{f'(x)}\left[\frac{f'(x)}{f(x)}\right] = \lim_{x\to\alpha}\frac{f(x)}{f'(x)}\left(\frac{f'(x)}{f(x)} - \left\{\frac{f'(x)}{f(x)}\right\}\right)$, where {.} denotes the fractional part function.

$$= \lim_{x\to\alpha} \frac{f(x)}{f'(x)} \times \frac{f'(x)}{f(x)} - \lim_{x\to\alpha} \frac{f(x)}{f'(x)} \left\{\frac{f'(x)}{f(x)}\right\}$$

$$1 - \lim_{x\to 0} \frac{x-\alpha}{2\phi(x)+(x-\alpha)\phi'(x)} \left\{\frac{2\phi(x)+(x-\alpha)\phi'(x)}{(x-\alpha)\,\phi(x)}\right\} = 1 - 0 = 1.$$
