DISCUSSION CLASS

1. A farmer purchased 100 head of live stock for a total cost of ₹4000/-. Prices were as follows : Calves @ ₹120., lambs @ ₹50, Pigs @ ₹25. If the farmer obtained atleast one animal of each type, how many of each did he buy?

$$120c + 50l + 25p = 4000 \Rightarrow 120c + 50l + 25(100 - c - l) = 4000$$

$$\Rightarrow \boxed{95c + 25l = 1500}$$

$$c + l + p = 100 \Rightarrow p = 100 - c - l$$

$(95, 25) = 5$

$95 = 25 \cdot 3 + 20$

$25 = 20(1) + 5$

$20 = 5 \cdot 4 + (0)$

$5 = 25 - 20(1)$

$= 25 - [95 - 3 \cdot (25)]$

$5 = 95(-1) + 25(4)$

$\Rightarrow 1500 = 95(-300) + 25(1200)$

$$\boxed{\begin{aligned} c_0 &= -300 \\ l_0 &= 1200 \end{aligned}}$$

$d=5,\ a=95,\ b=25,\ c_0=-300,\ l_0=1200$

$$\boxed{\begin{aligned} c &= -300+5t \\ l &= 1200-19t \end{aligned}}$$

Gen. solⁿ:

$$c = c_0 + \frac{b}{d}t$$

$$l = l_0 - \frac{a}{d}t$$

For +ve solⁿ: $-300+5t>0 \Rightarrow t>60$

$1200-19t>0 \Rightarrow t<\frac{1200}{19}=63.16$, $t\in\mathbb{Z}$

$\Rightarrow\ 60<t<63.16$

$t=61, 62, 63$

$t = 61,\quad c = -300 + 5\cdot 61 = 5$

$l = 1200 - 19\cdot 61 = 41$

$p = 100 - 5 - 41 = 54$

$t = 62,\quad c =$

$l =$

$p =$

$t = 63 : \{$

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2. Show that any integer of the form $6k+5$ is also of the form $3m+2$ but not conversely.

In D.A, put $a=k$, $b=3$.

$\therefore \exists$ unique integers q & r s.t $k = 3q + r$; $0 \le r < 3$.

$r = 0, 1, 2$.

$\therefore$ k can be of the form : $3q$, $3q+1$ or $3q+2$.

CASE-I: If $k = 3q$.

$$6k+5 = 6\cdot(3q)+5$$
$$= 6\cdot 3q + 3 + 2$$
$$= 3(6q+1) + 2$$
$$= 3m+2, \; m = 6q+1.$$

CASE-II : If $k = 3q+1$.

$$6k+5 = 6(3q+1)+5$$
$$= 6(3q)+11$$
$$= 6(3q)+9+2$$
$$= 3(6q+3)+2$$
$$= 3m+2 ; \; m = 6q+3.$$

CASE-III : $k = 3q+2$

$$6k+5 = 6(3q+2)+5$$
$$= 6(3q)+17$$
$$= 6(3q)+15+2$$
$$= 3(6q+5)+2$$
$$= 3m+2$$

3. Consider three positive real numbers a, b & c. Show that there cannot exist two distinct positive integers m & n such that both $a^m + b^m = c^m$ & $a^n + b^n = c^n$ holds. (ISI 2011)

Proof: Assume that: $n > m$.

$\hookrightarrow n - m > 0$

m & $n \in \mathbb{Z}^+$

$a, b, c \in \mathbb{R}^+$.

$$a^m + b^m = c^m;$$

$$\& \ a^n + b^n = c^n$$

$$c^n = c^{n-m} \cdot c^m$$

$$= c^{n-m} \cdot (a^m + b^m).$$

$$= c^{n-m} \cdot a^m + c^{n-m} \cdot b^m.$$

$$c^n > a^{n-m} \cdot a^m + b^{n-m} \cdot b^m = a^n + b^n = c^n.$$

$c^m > a^m \Rightarrow c > a.$

$c^m > b^m \Rightarrow c > b$

$c^{n-m} > a^{n-m}$.

$c^{n-m} > b^{n-m}$.

$c^n > a^n + b^n$.

Contradiction

$c^n > c^n$.

Assume that: $\boxed{m > n}$

$a^m + b^m = c^m \longrightarrow (1)$

$a^n + b^n = c^n$

$\left(\frac{a}{c}\right)^m + \left(\frac{b}{c}\right)^m = 1.$

$\left(\frac{a}{c}\right)^n + \left(\frac{b}{c}\right)^n = 1.$

$\because m > n.$

$\therefore \left(\frac{a}{c}\right)^m < \left(\frac{a}{c}\right)^n.$

$+ \quad \left(\frac{b}{c}\right)^m < \left(\frac{b}{c}\right)^n.$

$\left(\frac{a}{c}\right)^m + \left(\frac{b}{c}\right)^m < \left(\frac{a}{c}\right)^n + \left(\frac{b}{c}\right)^n$

$\Rightarrow 1 < 1 \rightarrow$ Contradiction.

$a < c. \Rightarrow \frac{a}{c} < 1$

$b < c. \Rightarrow \frac{b}{c} < 1.$

$a^x \rightarrow$ m. $\uparrow$ for $a > 1.$

$a^x \rightarrow$ m. $\downarrow$ for $0 < a < 1.$

$m > n$

$\Rightarrow a^m > a^n$ when $a > 1.$

$\Rightarrow a^m < a^n$ when $0 < a < 1.$

If $x > y$, then.

(i) $a^x > a^y$ if $a > 1.$

(ii) $a^x < a^y$ if $0 < a < 1.$

④ (i) The square of any integer is either of the form $3k$ or $3k+1$.

$a \to$ integer.

By D.A, $3k$, $3k+1$, $3k+2$.

Case-I: $a = 3k$, $a^2 = 9k^2 \Rightarrow a^2 = 3(3k^2) = 3K$.

Case-II: $a = 3k+1$, $a^2 = (3k+1)^2 \Rightarrow a^2 = 9k^2 + 6k + 1 = 3(3k^2+2) + 1$.

$= 3K + 1$.

Case-III: $a = 3k+2$, $a^2 = (3k+2)^2 \Rightarrow a^2 = 9k^2 + 12k + 4$

$= 9k^2 + 12k + 3 + 1$.

$= 3(3k^2 + 4k + 1) + 1$

$= 3K + 1$.

(ii) The cube of any integer is of the form : $9k$, $9k+1$ or $9k+8$.

Proof : By D.A, any integer a is of the form $3k$, $3k+1$, $3k+2$

Case-I : $a = 3k$,

then $a^3 = (3k)^3 = 27k^3 = 9(3k^3) = 9K$; $K = 3k^3$

Case-II : $a = 3k+1$,

then $a^3 = (3k+1)^3 \Rightarrow a^3 = 27k^3 + 27k^2 + 9k + 1$.

$= 9K + 1$.

Case-III : $a = 3k+2$.

then $a^3 = (3k+2)^3 = 27k^3 + 54k^2 + 36k + 8$.

$= 9K + 8$.

(iii) The fourth power of any integer is either of the form $5k$ or $5k+1$

Proof: By D.A., any integer n is of the form $5k, 5k+1, 5k+2, 5k+3, 5k+4$

Case-I: If $n = 5k$, $n^4 = (5k)^4 = 5(125k^4) = 5k$.

Case-II: If $n = 5k+1$, $n^4 = (5k+1)^4 = (\text{multiple of } 5) + 1$.

$= 5k + 1, k \in \mathbb{Z}$.

Case-III: If $n = 5k+2$, $n^4 = (5k+2)^4 = (\text{multiple of } 5) + 16$.

$16 = 15 + 1$.

$= 5k + 1$.

Case-IV: If $n = 5k+3$, $n^4 = (5k+3)^4 = (\text{multiple of } 5) + 81$

$81 = 80 + 1 = 5k + 1$

Case-V: If $n = 5k+4$, $n^4 =$ ______

5. Prove that no integer in the sequence

11, 111, 1111, 11111, ... is a perfect square.

By D.A, any integer is of the form $2k$ or $2k+1$.

Case-I: If $n = 2k$, $n^2 = (2k)^2 = 4k^2 = 4K$.

Case-II: If $n = 2k+1$, $n^2 = (2k+1)^2$

$\Rightarrow n^2 = 4k^2 + 4k + 1$

$\Rightarrow n^2 = 4K + 1$.

$\because$ Any integer in the given sequence is of the form $4k+3$, hence it is not a perfect square.

$11 = 8 + 3$
$= 4k + 3$.

$111 = 108 + 3$
$= 4k + 3$

$1111 = 1108 + 3$.
$= 4k + 3$.

$11111 = 11108 + 3$.
$= 4k + 3$.

$\vdots$

(6.) If n is an odd integer, show that $n^4 + 4n^2 + 11$ is of the form $16m$, where m is an integer.

Any integer n is of the form $\cancel{4k}$, $4k+1$, $\cancel{4k+2}$ or $4k+3$.

$(a+b)^4$
$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

Case-I : $n = 4k+1$.

$$n^4 + 4n^2 + 11 = (4k+1)^4 + 4(4k+1)^2 + 11$$
$$= 256k^4 + 256k^3 + 96k^2 + 16k + 1 + (64k^2 + 32k + 4) + 11$$
$$= 16k + 1 + 4 + 11 = 16k + 16 = 16(k+1)$$
$$= 16m.$$

Case-II : $n = 4k+3$,

$$n^4 + 4n^2 + 11 = (4k+3)^4 + 4(4k+3)^2 + 11$$

7. Prove that the difference of 2 consecutive cubes is never divisible by 2.

Sol^n^ : $(n+1)^3 - n^3 = \cancel{n^3} + 3n^2 + 3n + 1 - \cancel{n^3}$

$= 3n^2 + 3n + 1$

$= 3n(n+1) + 1$ → (A multiple of 2) + 1.

$\because 2 \mid n(n+1)$;

$\therefore$ $\therefore 2 \nmid (n+1)^3 - n^3$

8. Let $a, b, c \in \mathbb{N}$ be such that $a^2+b^2=c^2$ & $c-b=1$. Prove that

(i) a is odd (ii) b is div by 4 (iii) a^b+b^a is div. by c.

(ISI - 2018)

$a^2+b^2=c^2 \therefore a^2+b^2=(b+1)^2$.

$c-b=1$

$\hookrightarrow c=1+b$.

$\Rightarrow a^2+\cancel{b^2}=\cancel{b^2}+2b+1$.

$\Rightarrow \boxed{a^2=2b+1.}$

$\Rightarrow a^2 \in$ odd.

$\Rightarrow a \in$ odd.

$a^2+b^2=c^2 \Rightarrow a^2=c^2-b^2 \Rightarrow a^2=(c+b)(c-b) \Rightarrow \boxed{a^2=c+b} \longrightarrow (*)$

$c-b \in$ odd.

$\Rightarrow a^2 \in$ odd.

$\Rightarrow a \in$ odd.

$2b = a^2 - 1$

$\Rightarrow b = \dfrac{(a-1)(a+1)}{2}$.

Let $a = 2k+1$, $b = \dfrac{2k(2k+2)}{2} = 2k(k+1) = 4\lambda$.

$\Rightarrow b = 4\lambda$.

$\therefore a^2 \equiv 1 \pmod 8$.

$\boxed{n^2 \equiv 1 \pmod 8}$.

$\Rightarrow a^2 = \boxed{8k+1}$.

$\Rightarrow 8k + \cancel{1} = 2b + \cancel{1}$.

$\Rightarrow \overset{4}{\cancel{8}}k = \cancel{2}b \Rightarrow b = 4k$.

multiple of 4. | $4 \mid b$.

$$a^b = a^{4k} = (a^2)^{2k} \longrightarrow (2c-1)^{2k} \longrightarrow (-1)^{2k} = 1.$$

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$$a^2 = c+b \ \& \ c-b = 1.$$

$$2c = a^2 + 1.$$

$$\Rightarrow \boxed{a^2 = 2c-1}$$

$$b^a = (c-1)^a = (-1)^a = -1$$

$a^b + b^a$ is div. by c.